

# The Self-injective Cluster Tilted Algebras

Claus Michael Ringel

**Abstract.** We are going to determine all the self-injective cluster tilted algebras. All are of finite representation type and special biserial. There are two different classes. The first class are the serial (or Nakayama) algebras with  $n \geq 3$  simple modules and Loewy length  $n-1$ . The second class of algebras has an even number  $2m$  of simple modules, they are exhibited by quivers and relations at the end of this note.

Let  $k$  be a field. The algebras we will deal with are finite-dimensional associative  $k$ -algebras with 1. Given such an algebra  $A$ , the modules considered will usually be (finite-dimensional) left  $A$ -modules. The  $k$ -duality will be denoted by  $D = \text{Hom}(-, k)$ .

The class of cluster tilted algebras has been introduced by Buan, Marsh and Reiten [BMR]. According to Zhu [Z] and Assem, Brüstle and Schiffler [ABS], they can be defined as the semi-direct extension of  $A$  by the  $A$ - $A$ -bimodule  $\text{Ext}^2(D(A_A), {}_A A)$ , where  $A$  is a tilted algebra. The aim of this note is to single out those cluster tilted algebras which are self-injective. It is rather easy to see that self-injective cluster tilted algebras have to be of finite representation type. We will see that all are special biserial. There are two different classes. The first class are the serial (or Nakayama) algebras with  $n \geq 3$  simple modules and Kupisch series  $(n-1, \dots, n-1)$ . The second class of algebras has an even number  $2m$  of simple modules, they are exhibited by quivers and relations at the end of this note.

## First considerations.

Let  $H$  be a connected hereditary finite-dimensional  $k$ -algebra. Let  $n$  be the number of isomorphism classes of simple  $H$ -modules. We denote by  $\mathcal{C}(H)$  the corresponding cluster category; by definition, this is the orbit category of the derived category  $D^b(\text{mod } H)$  with respect to the functor  $F = \tau_d^{-1}[1]$ , where  $[1]$  is the shift functor and  $\tau_d$  the Auslander-Reiten translation functor in the derived category. We denote by  $\tau_c$  the Auslander-Reiten translation functor in the cluster category; both  $\tau_d$  and  $\tau_c$  are invertible. The Auslander-Reiten translation in  $\text{mod } H$  itself will be denoted by  $\tau_H$ .

Let  $T$  be a multiplicity-free tilting  $H$ -module, we may consider it as an object in the cluster category, and we denote by  $\tilde{A}$  its endomorphism ring in the cluster category. This is a typical cluster tilted algebra. Write  $T = \bigoplus_{i=1}^n T_i$  with  $T_i$  indecomposable.

The question which we consider here is: when is  $\tilde{A}$  self-injective? Note that  $\tilde{A}$  is self-injective if and only if  $\tau_c^2 T$  is isomorphic to  $T$ . Namely, the category  $\text{mod } \tilde{A}$  is the factor category of  $\mathcal{C}(H)$  modulo the ideal generated by the objects  $\tau_c T_i$ ; in this factor category, the objects  $T_i$  are the indecomposable projective ones, the objects  $\tau_c^2 T_i$  the indecomposable injective ones. In order for  $\tilde{A}$  to be self-injective we just need that any indecomposable injective  $\tilde{A}$ -module (thus any  $\tau_c^2 T_i$ ) is projective (thus isomorphic to some  $T_j$ ).

### Exclusion.

Assume that  $T$  and  $\tau_c^2 T$  are isomorphic. Now  $T$  has only finitely many indecomposable direct summands (namely  $n$ ); this means that any object  $T_i$  has to be  $\tau_c$ -periodic. If  $H$  is wild, then  $\mathcal{C}(H)$  has no  $\tau_c$ -periodic objects at all. Thus  $H$  cannot be wild. If  $H$  is tame, then the  $\tau_c$ -periodic objects in  $\mathcal{C}(H)$  are actually  $\tau_H$ -periodic modules. But it is well-known that the number of  $\tau_H$ -periodic indecomposable direct summands of a tilting  $H$ -module is at most  $n - 2$ . (A stable tube of rank  $r$  can have only  $r - 1$  indecomposable direct summands which are pairwise Ext-orthogonal, and the sum of these numbers  $r - 1$  over all the tubes is  $n - 2$ , see [DR].) This shows that  $H$  has to be representation-finite.

Note that the  $\tau_c$ -orbit of any summand  $T_i$  has to have an even number of elements, so that half of these elements can be direct summands of  $T$ .

We have to consider now the various cases  $A_n, B_n, \dots, E_8, G_2$  in detail. Let  $\Delta$  be the quiver of  $H$  (we consider it as a valued quiver (see [DR]), if we deal with one of the cases  $B_n, C_n, F_4$  and  $G_2$ ). Let  $a$  be a vertex of  $\Delta$ . Denote by  $P(a)$  the indecomposable projective  $H$ -module corresponding to the vertex  $a$ . We are looking for natural numbers  $t$  such that  $\text{Hom}(P(a), \tau_H^{-t} P(a)) \neq 0$ . If  $\tau_H^{-t} P(a)$  is not injective, then we see that  $\text{Ext}^1(\tau_H^{-t-1} P(a), P(a)) \neq 0$ . Now, if  $t$  is odd, then  $-t - 1$  is even: We conclude that in this case neither  $P(a)$  nor any other element of the  $\tau_c$ -orbit of  $P(a)$  can be a direct summand of  $T$ . In order to decide whether  $\text{Hom}(P(a), \tau_H^{-t} P(a))$  is zero or not, one just has to calculate the hammock function starting at  $P(a)$ , see [G] or [RV].

Let us assume now that  $T$  has an indecomposable direct summand  $T_i$  which is in the  $\tau_H$ -orbit of  $P(a)$ .

(1) *The vertex  $a$  is a boundary vertex of  $\Delta$ .* Assume for the contrary that  $a$  is an interior vertex. We have  $\text{Hom}_H(P(a), \tau^{-1} P(a)) \neq 0$ . If  $\Delta$  is different from  $A_3$ , then  $\tau^{-1} P(a)$  is not injective, thus we get a contradiction. If  $\Delta$  is of type  $A_3$ , then the orbit of  $\tau_c$  corresponding to  $a$  has length 3, thus is of odd length, again a contradiction.

(2) *The edge between  $a$  and its (unique) neighbor has valuation  $(1, 1)$ .* Here again we see that otherwise  $\text{Hom}_H(P(a), \tau^{-1} P(a)) \neq 0$ , and in this case  $\tau^{-1} P(a)$  cannot be injective. This immediately excludes  $G_2$ . Also, together with (1) it excludes  $B_n$  and  $C_n$ : Namely, all the indecomposable summands of  $T$  would have to belong to a single  $\tau_H$ -orbit — but since the simple  $H$ -modules do have two different kinds of endomorphism rings, the same is true for any tilting  $H$ -module ([R], alternatively, we also could argue that the orbit in question has not enough elements).

Assume that  $\Delta$  is of type  $D_n$  or  $E_n$ , and that  $a = a_1, a_2, \dots, a_p$  is the minimal path from  $a$  to the branching vertex  $a_p$ . If necessary, we write  $p = p(a)$ .

(3) *The number  $p$  is even.* (We have  $\text{Hom}_H(P(a), \tau^{-p} P(a)) \neq 0$ , and  $\tau^{-p} P(a)$  is not injective).

(4) *If  $\Delta = E_n$ , then  $p > 2$ .* (Again,  $\text{Hom}_H(P(a), \tau^{-3} P(a)) \neq 0$ , and  $\tau^{-3} P(a)$  is not injective).

The assertions (1), (3) and (4) exclude immediately the cases  $E_6, E_8$ , and they show that in case  $E_7$  all the indecomposable direct summands of  $T$  have to belong to the  $\tau_c$ -orbit which contains the module  $P(a)$  with  $p(a) = 4$ . But any  $\tau_c$ -orbit of  $\mathcal{C}(H)$  with  $H$  of type  $E_7$  is of length 10, thus only 5 indecomposable direct summands of  $T$  can belong to this orbit. Thus also  $E_7$  is excluded. Similarly, we see that in case  $H$  is of type  $D_n$ , and  $a(p) > 2$ ,

then  $n = a(p) + 2$  has to be even.

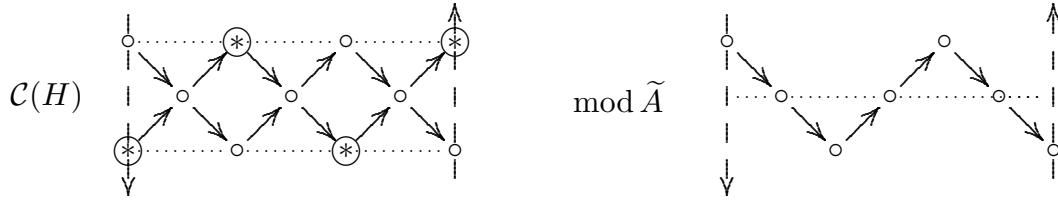
In order to exclude the case  $F_4$ , we only have to notice that  $\text{Hom}(P(a), \tau^{-3}P(a)) \neq 0$  for any of the two boundary vertices.

We still have to consider the cases  $A_n$ . The two  $\tau_H$ -orbits of  $H$  yield a single  $\tau_c$ -orbit containing  $n + 3$  elements. These are the only possible summands of  $T$ , thus we must have  $\frac{1}{2}(n + 3) = n$ , therefore  $n = 3$ .

Besides  $A_3$  also the cases  $D_n$  remain. Assume we are in case  $D_n$  and  $p(a) > 2$ . The  $\tau_c$ -orbit of  $P(a)$  contains precisely  $n$  elements, this shows that  $n$  has to be even. For  $n \geq 5$ , there are two  $\tau_H$ -orbits  $a$  with  $a(p) = 2$ ; for  $n = 4$ , there are three such  $\tau_H$ -orbits. In case  $n$  is odd, the two  $\tau_H$ -orbits with  $a(p) = 2$  combine to form part of a single  $\tau_c$ -orbit with  $2n$  elements. In case  $n$  is even, the  $\tau_H$ -orbits with  $a(p) = 2$  are contained in two (or, for  $n = 4$ , three) separate  $\tau_c$ -orbits, each having  $n$  elements. All these  $\tau_c$ -orbits actually occur, as we are going to show next.

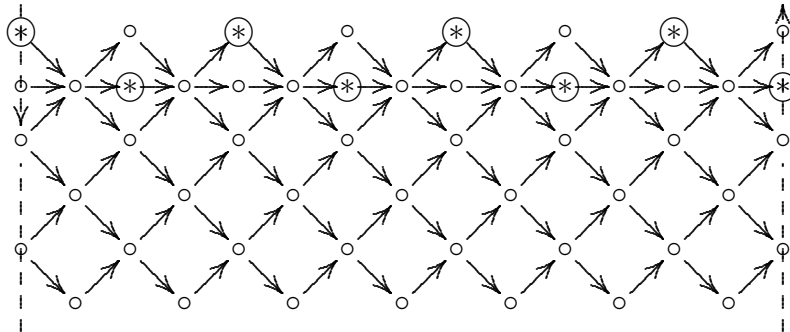
### The possible cases.

**1. Case  $A_3$ .** The algebra  $\tilde{A}$  is serial with 3 simple modules and Kupisch series  $(2, 2, 2)$ . Here are the Auslander-Reiten quivers of  $\mathcal{C}(H)$  and of  $\text{mod } \tilde{A}$ . Note that in both cases the left hand boundary has to be identified with the right hand side using a twist in order to form a Möbius strip. The summands  $T_i$  are marked by a star  $*$ .



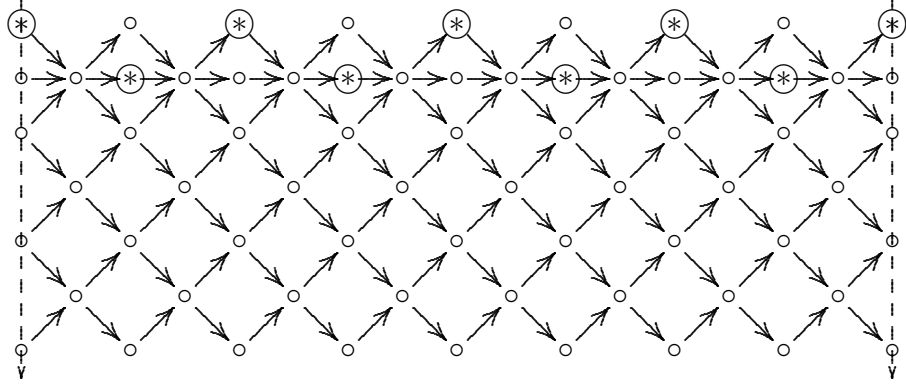
**2. The case  $D_n$  with  $n \geq 4$ , using two short arms.** Let us present the cases  $n = 7$  and  $n = 8$ .

First, we present the case  $n = 7$ . In this case, the upper part (given by the two short arms) provides a Möbius strip (the lower part given by the long arm is always a cylinder). In general, for  $n$  odd one, the short arm part of the Auslander-Reiten quiver is a Möbius strip.



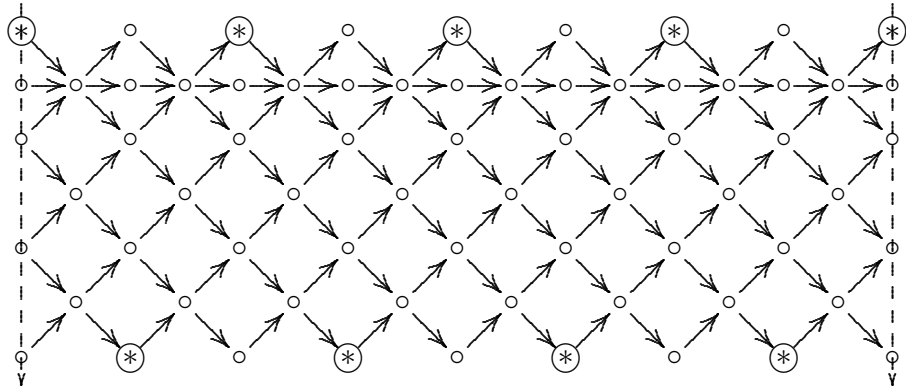
Here is the case  $n = 8$ . Now the left hand boundary has to be identified with the right hand side without any twist. In general, for  $n$  even, the Auslander-Reiten quiver of  $\mathcal{C}(H)$

is just  $\mathbb{Z}D_n/\tau^n$ .

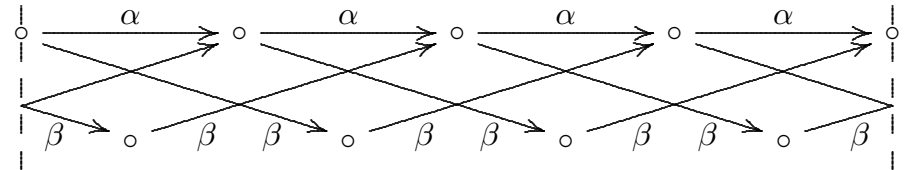


Always, we obtain a serial algebra with  $n$  simple modules and the Kupisch series is  $(n-1, \dots, n-1)$ . As a quiver with relations, it is given by a cycle with  $n$  vertices and  $n$  arrows. If we label all the arrows by  $\alpha$ , then the relations can be written just in the form  $\alpha^n = 0$ . The only difference between the cases  $n$  even or odd concerns the Nakayama permutation: In the even case, it has two orbits, in the odd case only one. Note that the  $A_3$  case may be considered as a  $D_3$ -case, thus as part of the sequence of serial algebras.

**3. The case  $D_{2m}$  with  $m \geq 3$ , using the long arm and one short arm.** Here we present the case  $D_{2m}$  with  $m = 4$ .



The quiver of  $\tilde{A}$  has the following shape (again, the right hand side has to be identified with the left hand side):



with relations

$$\alpha\beta = \beta\alpha = 0, \quad \alpha^{m-1} = \beta^2.$$

## References

- [ABS] Assem, Brüstle, Schiffler: Cluster-tilted algebras as trivial extensions.  
arXiv: math.RT/0601537
- [BMR] Buan, Marsh, Reiten: Cluster tilted algebras. Trans. Amer. Math. Soc. (To appear).  
arXiv: math.RT/0402075
- [DR] Dlab, Ringel: Indecomposable representations of graphs and algebras. Mem. Amer. Math. Soc. 173 (1976).
- [HR] Happel, Ringel: Tilted algebras. Trans. Amer. Math. Soc. 274 (1982), 399-443.
- [KR] Keller, Reiten: Cluster-tilted algebras are Gorenstein and stably Calabi-Yau. arXiv: math.RT/0512471
- [R] Ringel: Exceptional modules are tree modules. Lin. Alg. Appl. 275-276 (1998) 471-493.
- [RV] Ringel, Vossieck: Hammocks. Proc. London Math. Soc. (3) 54 (1987), 216-246.
- [Z] Bin Zhu: Equivalences between cluster categories. (To appear).  
arXiv:math.RT/0511382.